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ANALYSIS OF LIQUID DROPLET DEFORMATION IN A GAS FLOW

V. V. Voronin

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The small perturbation method is used to obtain equations describing the dynamics of a liquid droplet in a flow of ideal incompressible gas. The stability criteria and droplet disintegration time are determined.

The principles of motion of a liquid droplet in a gas flow with some relative velocity are of great practical interest and have been actively studied for several decades. The present state of studies of liquid droplet interaction with a carrier flow is presented quite fully in the review [1]. In particular, analysis of numerous experimental data has established a qualitative classification of the main types of droplet disintegration in a gas flow, which develops upon increase of the Weber number We; at We $\leq$ 100 droplet breakup is preceded by a "parachute"-type deformation which can be described within the framework of the approximate theory of flow over a deformed body.

Linearization of the defining equations establishes that a droplet in a gas flow spreads in the transverse direction with the form of the flattened droplet being close to an ellipsoid of rotation. Spreading of the droplet, which is maintained in the process of deformation of the ellipsoid form, was studied in detail in a number of works [2-5], in which simple asymptotes were obtained for the transverse deformation together with stability criteria for the droplet.

An analytical method for calculation of nonsteady-state motion and spreading of plane and axisymmetric drops of a viscous liquid in a gas flow was developed in [6]. This method is based on expansion of the Navier—Stokes equation in a small parameter, while the boundary problem is reduced to solution of an infinite system of differential equations with constant coefficients. In the absence of viscosity the system contains a finite number of equations

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and permits an exact solution, which allows study of the spreading of a drop of ideal liquid with consideration of surface tension forces. A unique feature of the model is the use of an experimental pressure profile on the droplet boundary in analyzing the equations of liquid motion within the drop.

Below we will present a solution of the problem of motion and deformation of a liquid drop in an ideal incompressible liquid within the framework of potential theory. Analysis of the kinematic and dynamic conditions will produce an infinite system of related quasilinear differential equations in the velocity and amplitudes of spherical harmonics of drop deformation. The solution will be studied asymptotically.

We will consider the deformation of an initially spherical drop of ideal incompressible liquid with density ρ_1, moving with a velocity U(t) in an ideal incompressible gas with density ρ_0. We will use a relative system of spherical coordinates (r, θ, ψ), the origin of which is attached to the moving center of gravity of the drop. The flow has axial symmetry, and the potentials of flows inside and outside the drop can be represented in the form of the sum of particular solutions of the Laplace equation for the internal and external regions:

$$\Phi_0(r, \theta, t) = \varphi_0 + \varphi_1 = -U\frac{R^3}{2r^2}\cos\theta + \sum_{n=1}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta), \; r > R;$$
$$\Phi_1(r, \theta, t) = \sum_{n=2}^{\infty}A_n r^n P_n(\cos\theta), \; r \leqslant R, \tag{1}$$

where R is the droplet radius. The flow potential Φ_0 consists of the main component φ_0, related to rectilinear motion of the sphere within the gas, and the potential φ_1, describing the perturbation of the flow produced by droplet deformation: the flow potential Φ_1 is related only to droplet deformation.

The boundary of the drop is a spherical surface with regular perturbation $\xi(\theta, t)$ imposed thereon: the function $F(r, \theta, t)$ which specifies the drop surface can be written as a series of spherical harmonics

$$F(r, \theta, t) = r - R(t) - \xi(\theta, t) = r - R - \sum_{n=2}^{\infty}\xi_n P_n(\cos\theta) = 0. \tag{2}$$

On the droplet surface the kinematic condition of nonpenetration of liquid through the boundary is satisfied, and can be written in the form

$$\frac{\partial F}{\partial t} + (\nabla\Phi_i - \mathbf{U}\delta_{0i})\nabla F = 0, \; i = 0,1 \text{ at } F(r, \theta, t) = 0, \tag{3}$$

where $\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}\right)$ is a differential operator and δ_{nk} is the Kronecker symbol ($\delta_{nk} = 1$, if $n = k$; $\delta_{nk} = 0$ if $n \neq k$); the velocity $\mathbf{U} = U\cdot(\cos\theta, -\sin\theta)$.

In accordance with the methods of small perturbation theory we assume that $\xi \ll R$ and $\varphi_1, \Phi_1 \ll \varphi_0$; then from the condition of constancy of drop volume it follows that in the approximation linear in ξ/R, $R(t) = R_0$.

The functions and their derivatives in Eq. (3) are expanded in Taylor series in the vicinity of the unperturbed drop boundary $r = R_0$. After dropping terms of second and higher order of smallness in ξ/R, we obtain the linearized kinematic equations

$$\frac{\partial\xi}{\partial t} - \frac{\partial\Phi_1}{\partial r} = 0$$
$$\text{at } r = R_0. \tag{4}$$
$$\frac{\partial\xi}{\partial t} - \frac{\partial\varphi_1}{\partial r} + \nabla\xi\cdot(\nabla\varphi_0 - \mathbf{U}) - \xi\frac{\partial^2\varphi_0}{\partial r^2} = 0$$

We substitute in Eq. (4) expanded expressions for φ_0, φ_1, Φ_1, and ξ; the equations are multiplied by $P_j(\cos\theta)\sin\theta$ and integrated over the angle θ with limits $[0, \pi]$. The subscript j is then assigned integer values; with consideration of the orthogonality of Legendre polynomials, we obtain a system of relationships

$$A_n = R_0^{-n}\frac{\dot\xi_n}{n},\quad n=1,\ 2,\ 3,\ \ldots$$

$$B_n = R_0^{n+2}\left[-\frac{\dot\xi_n}{n+1}+\frac{3nU}{2R_0}\left(\frac{\xi_{n+1}}{2n+3}-\frac{\xi_{n-1}}{2n-1}\right)\right]. \quad (5)$$

It should be noted that the second relationship of system (5) coincides exactly with the result of [7], obtained by a more complex and cumbersome method.

On the drop surface there is a pressure discontinuity upon transit of the boundary along the normal thereto, caused by surface tension forces. In mathematical form this condition can be written as

$$\rho_0\left[\frac{\partial\Phi_0}{\partial t}-\mathbf{U}\nabla\Phi_0+\frac{1}{2}(\nabla\Phi_0)^2\right]-\rho_1\left[\frac{\partial\Phi_1}{\partial t}+\frac{1}{2}(\nabla\Phi_1)^2\right]=\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\ \text{at}\ F(r,\ \theta,\ t)=0. \quad (6)$$

Dynamic condition (6) can be linearized and reduces to the form

$$\rho_0\left\{\frac{\partial\varphi_0}{\partial t}-\mathbf{U}\nabla\varphi_0+\frac{1}{2}(\nabla\varphi_0)^2+\frac{\partial\varphi_1}{\partial t}-\mathbf{U}\nabla\varphi_1+\nabla\varphi_0\nabla\varphi_1+\right.$$
$$\left.+\xi\frac{\partial}{\partial r}\left[\frac{\partial\varphi_0}{\partial t}-\mathbf{U}\nabla\varphi_0+\frac{1}{2}(\nabla\varphi_0)^2\right]\right\}-\rho_1\frac{\partial\Phi_1}{\partial t}=\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\ \text{at}\ r=R_0. \quad (7)$$

Equation (7) is multiplied by $P_j(\cos\theta)\sin\theta$ and integrated over angle θ with limits $[0, \pi]$. By subsequent selection of $j = n$ we obtain a system of related differential equations for ξ_n:

$$2\ddot\xi_n\frac{n+n\nu+\nu}{n(n+1)}+\frac{9}{2}u^2\xi_n\left[1-n\frac{(n-1)^2}{4n^2-1}-\frac{4n^3+6n^2-1}{(2n+3)(4n^2-1)}-\right.$$
$$\left.-\frac{(n+1)^2(n+2)}{(2n+1)(2n+3)}\right]+3u\left(\dot\xi_{n-1}-\frac{2n+1}{2n+3}\dot\xi_{n+1}\right)-\dot u\left(\frac{5n+2}{2n+3}\xi_{n+1}-\right.$$
$$\left.-\frac{n}{2n-1}\xi_{n-1}\right)+\frac{9}{2}u^2\left[\frac{n(n+1)(n+2)}{(2n+3)(2n+5)}\xi_{n+2}+\right.$$
$$\left.+\frac{n(n-1)(n-2)}{(2n-1)(2n-3)}\xi_{n-2}\right]+4\frac{n^2+n+2}{\mathrm{We}}\xi_n+\frac{3}{2}u^2\delta_{n2}=0,\ n=2,\ 3,\ 4,\ldots \quad (8)$$

Equation (8) is presented in dimensionless form, dedimensionalized with the factor droplet radius R_0, initial droplet velocity U_0, and gas density ρ_0. Two parameters then appear in dynamic equation (8), the density ratio $\nu = \rho_1/\rho_0$, and the Weber number $\mathrm{We} = 2\rho_0 U_0^2 R_0/\sigma$.

In an ideal gas the droplet braking equation is obtained from the theorem of momentum for a deformed body [8], i.e.,

$$m\frac{dU}{dt}=-\rho_0\frac{d}{dt}\int_S\Phi_0\cdot(\mathbf{n}\cdot\mathbf{i})\,dS=$$
$$=-2\pi R_0^2\rho_0\frac{d}{dt}\int_0^\pi\Phi_0\cdot(\cos\theta+2\xi\cos\theta+\xi'\sin\theta)\cdot\sin\theta d\theta. \quad (9)$$

Substituting in Eq. (9) the droplet mass $m = {}^4/_3\pi R_0^3\rho_1$ and the expression for the potential Φ_0, we obtain the dimensionless braking equation in the form

$$\dot u\left(2\nu+1-\frac{9}{5}\xi_2\right)-\frac{9}{5}u\dot\xi_2=0. \quad (10)$$

Equations (8) and (10) completely describe the dynamics of motion of a deformed droplet of ideal liquid in an ideal gas. The Cauchy problem for this system of equations is defined by the initial conditions

$$u(0) = 1;\ \xi_n(0) = \dot{\xi}_n(0) = 0,\ n = 2,\ 3,\ \ldots, \tag{11}$$

and the values of the parameters ν and We. Equation (8) consists of differential equations linear in the small perturbations ξ_n, the coefficients of which contain the velocity of droplet motion. But since the velocity u(t) is defined by Eq. (10) in terms of droplet deformation, the system of equations (8), (10) is nonlinear. An exact solution of the system can be obtained by computer iteration methods.

We will consider the limiting case in which the density ratio $\nu \gg 1$. Equation (8) at n = 2 and Eq. (10) with initial conditions (11) can be written approximately in the form

$$2\dot{u}\nu - \frac{9}{5}\dot{u}\xi_2 - \frac{9}{5}u\dot{\xi}_2 = 0, \tag{12}$$

$$7\nu\ddot{\xi}_2 + 4\xi_2\left(\frac{56}{\mathrm{We}} - \frac{27}{5}u^2\right) + \frac{21}{2}u^2 = 0. \tag{13}$$

We seek a solution for u(t) and $\xi_2(t)$ in the form of expansions in the small parameter ν^{-1}:

$$\xi_2 = \xi_2^{(0)} + \varepsilon_1(\nu^{-1})\xi_2^{(1)} + \ldots;\quad u(t) = 1 + \delta_1(\nu^{-1})u_1 + \ldots$$

After substitution of these expansions in Eq. (13) and dropping of smaller terms, we obtain an ordinary differential equation for $\xi_0^{(2)}$ with solution

$$\xi_2^{(0)} = D(\exp(\lambda t) + \exp(-\lambda t) - 2);$$
$$\lambda^2 = \frac{4}{7\nu}\left(\frac{27}{5} - \frac{56}{\mathrm{We}}\right);\ D = -\frac{3}{4}\frac{1}{\nu\lambda^2}. \tag{14}$$

For $\lambda^2 < 0$ there is a pulsating motion of the drop boundary relative to the unperturbed position; for $\lambda^2 > 0$ the perturbation increases, finally leading to droplet breakup. From the condition $\lambda^2 = 0$ we find the critical value of the Weber number We*, which defines the stability limit for the drop with respect to action of an air flow:

$$\mathrm{We}^* = \frac{280}{27} \simeq 10.4, \tag{15}$$

which agrees well with the experimental value $\mathrm{We}_e^* = 12$-14 presented in [9]. It should be noted that the value of We* obtained in Eq. (15) is independent of the density ratio ν and refines earlier estimates [2, 5, 6].

Of special interest is the case of high Weber numbers We > We*, which corresponds to the increasing perturbation regime. For $\lambda t \ll 1$, solution (14) can be expanded in a Taylor series and after dropping small terms we obtain $\xi_2^{(0)} = -3t^2/4\nu$. This approximate value of $\xi_2^{(0)}$ is substituted in Eq. (12) together with the expansion for the velocity u(t); equation of like terms leads to the explicit form of the function $\delta_1 = \nu^{-2}$. Consequently, in motion of a liquid drop in air it can be assumed that the drop velocity is constant to the accuracy of $O(\nu^{-2})$, i.e., u(t) = 1.

Analysis of the structure of Eq. (8) reveals that perturbations ξ_n, n = 3, 4, ..., are of order ν^{-2} and higher. Consequently, in the approximation linear in ν^{-1} deformations of the droplet surface are described by the equation

$$\xi(\theta,\ t) = -\frac{3t^2}{4\nu}P_2(\cos\theta). \tag{16}$$

Equation (16) was obtained with the assumption that $\xi(\theta, t) \ll 1$. However, extrapolation of solution (16) to the region of large deformations permits an estimate of droplet disintegration times which are close to experimental values.

The time interval from the moment of introduction of the liquid drop into the gas flow to the moment of contact of the upwind and downwind critical points on the droplet surface, corresponding to the disintegration time τ, is defined by the equation

$$2 + \xi(0,\ \tau) + \xi(\pi,\ \tau) = 0. \tag{17}$$

Substituting Eq. (16) in Eq. (17), we obtain a solution for the disintegration time $\tau = 2\sqrt{\nu/3}$, or, in dimensionless form,

$$\tau = \frac{\sqrt{3}}{3}\frac{d}{U_0}\sqrt{\frac{\rho_1}{\rho_0}},\ d = 2R_0. \tag{18}$$

Solution (18) coincides with known estimates of droplet disintegration time in a gas flow [1, 2, 10].

NOTATION

F, function describing droplet surface; P_n, first-order Legendre polynomials of the n-th sort; m, droplet mass; R_0, droplet radius; R_1, R_2, major radii of curvature of droplet surface; (r, θ, ψ), coordinate system; U, U_0, u, droplet motion velocity, initial value, and dimensionless velocity; We, We*, Weber number and critical value thereof; ρ_1, ρ_0, ν, liquid and gas densities, ratio thereof; Φ_1, Φ_0, flow potentials within and without droplet; φ, flow potential for motion of a sphere; ξ, ξ_n, perturbation of droplet boundary, amplitude of spherical harmonics; S, droplet surface area; τ, droplet disintegration time; σ, surface tension coefficient; $\mathbf{n}$, vector of normal to droplet surface; $\mathbf{i}$, unit vector in direction of droplet motion.

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